Lemma (Taylor coefficients are controlled by size of function)

Let $f(z)$ be an analytic function on the disk $|z| \leq R$ such that $f(0)=0$, with Taylor expansion $f(z) = \sum_{n \geq 1} c_n z^n$.

Then, for $n \geq 1$, $|c_n| \leq 4 \max_{|z|=R} \frac{|Re(f(z))|}{R^n}$.

Proof: Let $z = Re^{2\pi i \theta}$, $\theta \in \mathbb{R}$.

Let $g(\theta) = Re(f(Re^{2\pi i \theta}))$, so $g(\theta) = g(\theta+1)$.

Say $c_n = a_n + i b_n$, for $a_n, b_n \in \mathbb{R}$.

We have $g(\theta) = Re(f(Re^{2\pi i \theta})) = Re\left(\sum_{n=1}^{\infty} c_n R^n e^{2\pi i n \theta}\right)$

$$= \sum_{n=1}^{\infty} R^n a_n \cos(2\pi n \theta) - \sum_{n=1}^{\infty} R^n b_n \sin(2\pi n \theta)$$

On the other hand, from Fourier inversion,

$$R^n a_n = 2 \int_0^1 g(\theta) \cos(2\pi n \theta)\, d\theta$$

$$R^n b_n = -2 \int_0^1 g(\theta) \sin(2\pi n \theta)\, d\theta.$$

Therefore, $|C_n| \le 2\max(|a_n|, |b_n|) \le \frac{4}{R^n} \max_{\theta\in[0,1]} |g(\theta)|$

$$= \frac{4}{R^n} \max_{|z|=R} \mathrm{Re}(f(z)). \qquad \square$$

Note that if $P(z) = C \prod_{j=1}^{k} (z - z_k)$ a polynomial, then $\frac{P'(z)}{P(z)} = \sum_{j=1}^{k} \frac{1}{(z-z_k)}$.

We want to generalise this to analytic functions f and approximate $\frac{f'}{f}$ as a sum over the zeros.

<u>Lemma</u> (Partial fraction approximation for analytic functions)

Let $f(z)$ analytic on the disk $|z| \le R$ with $f(0) \neq 0$. Let $z_1, \ldots, z_k \in \mathbb{C}$ denote the zeros of f inside the disk $|z| \le R/2$, listed with multiplicity.

Then for $|z| \le \frac{9R}{20}$, $\left| \frac{f'(z)}{f(z)} - \sum_{j=1}^{k} \frac{1}{z - z_j} \right| < c \frac{1}{R} \max_{|z|=R} \log \left| \frac{f(z)}{f(0)} \right|$

<u>Proof</u>: Let $g(z) = \frac{f(z)}{\prod_{j=1}^{k} (z - z_j)}$.

Then $g(z)$ has no zeros in $|z| \le R/2$ and analytic in this region.

Let $G(z) = \log\left(\frac{g(z)}{g(0)}\right)$, analytic in $|z| \le R/2$.

(say $G(z) = \int_0^z \frac{g'(s)}{g(s)}\,ds$, straight line contour 0 to z).

Then $G'(z) = \frac{f'(z)}{f(z)} - \sum_{j=1}^{K} \frac{1}{z - z_j}$.

Write $G(z) = \sum_{n=1}^{\infty} c_n z^n$ (valid for $|z| \le R/2$)

From previous lemma,

$$|c_n| \le \frac{4}{(R/2)^n} \max_{|z| = \frac{R}{2}} \mathrm{Re}(G(z))$$

$$= \frac{4}{(R/2)^n} \max_{|z| = \frac{R}{2}} \log\left|\frac{g(z)}{g(0)}\right|$$

From maximum modulus principle,

$$\max_{|z| = \frac{R}{2}} \left|\frac{g(z)}{g(0)}\right| \le \max_{|z| = R} \left|\frac{g(z)}{g(0)}\right| = \max_{|z| = R} \left|\frac{f(z)}{f(0)} \prod_{j=1}^{K} \frac{z_j}{z - z_j}\right|$$

Since $|z_j| \le R/2$, for all j, $\frac{|z_j|}{|z - z_j|} \le 1$ for $|z| = R$.

Thus $|c_n| \le \frac{4}{(R/2)^n} \max_{|z| = R} \log\left|\frac{f(z)}{f(0)}\right|$.

Finally, for $|z| \le \frac{9R}{20}$,

$$\left|\frac{f'(z)}{f(z)} - \sum_{j=1}^{k}\frac{1}{z-z_j}\right| = |G'(z)| = \left|\sum_{n=1}^{\infty} c_n n z^{n-1}\right|$$

$$\leq \frac{8}{R}\left(\max_{|z|=R} \log\left|\frac{f(z)}{f(0)}\right|\right) \underbrace{\sum_{n=1}^{\infty} n\left(\frac{9}{10}\right)^{n-1}}_{\text{converges}}$$

$$\ll \frac{1}{R} \max_{|z|=R} \log\left|\frac{f(z)}{f(0)}\right|.$$

$\square$

Notation: It is standard to denote the zeros of a complex function by ρ. By "$\sum_\rho$" or "$\prod_\rho$" we mean the sum/product over all zeros, unless specified differently.

When we specialise to $\xi(s)$, we let "$\sum_\rho / \prod_\rho$" denote the sum/product over the non-trivial zeros.!

Lemma: (Partial fraction expansion for $\xi(s)$).
Let $s = \sigma + it$, with $-1 \leq \sigma \leq 2$.
Then $$\frac{\xi'(s)}{\xi(s)} = -\frac{1}{s-1} + \sum_{\rho : |\rho - s| \leq 1} \frac{1}{s-\rho} + O(\log(2+|t|)).$$

(here the sum is over non-trivial zeros ρ of $\xi(s)$ with each zero of multiplicity m occurring m times)

Proof: Note that $\frac{\zeta'}{\zeta}(s)$ only has simple poles at the zeros of $\zeta(s)$ and at 1.

We first observe result follows for $|t| \leq 10$, because $\frac{\zeta'}{\zeta}(s)$ is $O(1)$ unless close to one of the finite number of poles, in which case

$$\frac{\zeta'(s)}{\zeta(s)} = -\frac{1}{s-1} + O(1), \text{ if } s \text{ is close to } 1.$$

Note indeed that since $\zeta(s) = \frac{1}{s-1} + O(1)$, then $\zeta'(s) = -\frac{1}{(s-1)^2} + O(1)$ (think Laurent expansion of $\zeta'(s)$ around 1, and conclusion follows).

Similarly, $\frac{\zeta'(s)}{\zeta(s)} = \frac{m_\rho}{s-\rho} + O(1)$ if s close to a zero ρ.

multiplicity of zero ρ.

Assume $|t| \geq 10$.

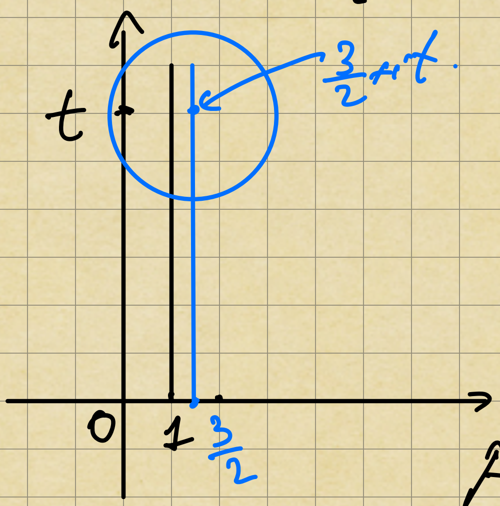
Let $g(z) = \zeta\left(\frac{3}{2} + z + it\right)$ and $R = 4$, so $g(0) \neq 0$ and $|g(z)| \ll (1+|t|)^4$ for $|z| \leq 4$.

(Same as last time)

Applying the previous lemma:

$$\frac{g'(z)}{g(z)} = \sum_{|\rho-\frac{3}{2}-it|<2} \frac{1}{z-\rho+\frac{3}{2}+it} + O(\log(2+|t|)),$$

for $|z| \le \frac{9}{5}$.



Let $z = \sigma - \frac{3}{2}$, so $g(z) = \zeta(\sigma+it)$,

and note $|z| \le \frac{9}{5}$ if $-1 \le \sigma \le 2$.

Also: $\{\rho : |\rho-\sigma-it| \le 1\} \subset \{\rho : |\rho-\frac{3}{2}-it|<2\}$

(when $\sigma \in [-1, 2]$).

Note that each zero ρ for which $|\rho-\sigma-it|>1$ contributes to the sum at most 1,

and from last time $|\{\rho : |\rho-\frac{3}{2}-it|\}| \ll \log(2+|t|)$

Hence
$$\left| \sum_{\substack{\rho:\, |\rho-\sigma-it|>1 \\ |\rho-\frac{3}{2}-it|<2}} \frac{1}{\sigma+it-\rho} \right| \ll \log(2+|t|).$$

Therefore
$$\frac{g'(z)}{g(z)} = \frac{\zeta'(\sigma+it)}{\zeta(\sigma+it)} =$$

$$= \sum_{\rho:\, |\rho-\sigma-it| \le 1} \frac{1}{\sigma+it-\rho} + \underbrace{\sum_{\substack{\rho:\, |\rho-\sigma-it|>1 \\ |\rho-\frac{3}{2}-it|<2}} \frac{1}{\sigma+it-\rho}}_{O(\log|t|)} + O(\log(2+|t|))$$

$\square$

Corollary: (Size of $\frac{\zeta'}{\zeta}(s)$ controlled away from zeros)

Let $s = \sigma + it$ with $\sigma \geq -1$. If the distance from s to the nearest zero of $\zeta(s)$ is at least $\gg \frac{1}{\log(2+|t|)}$, then $\frac{\zeta'}{\zeta}(s) = O\left((\log(2+|t|))^2\right)$.

Proof: Follows easily from last lemma and that $|\{\rho : |\sigma+it-\rho| \leq 2\}| \ll \log(2+|t|)$.

$\square$

Recall: • $\psi(x) = \sum_{n \leq x} \Lambda(n)$

• PNT $\iff \psi(x) \sim x$.

• For $\mathrm{Re}(s) > 1$, we have $-\frac{\zeta'}{\zeta}(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}$

Truncated Perron: $c = 1 + \frac{1}{\log x}$, $2 \leq T \leq 2x$

$$\psi(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(-\frac{\zeta'}{\zeta}(s)\right) \cdot x^s \frac{ds}{s} + O\left(\frac{x(\log x)^2}{T}\right).$$

Next we prove an explicit formula for $\psi(x)$:

Theorem (Explicit formula for $\psi(x)$)

Let $2 \le T \le 2X$. Then

$$\psi(X) = X - \sum_{|\mathrm{Im}\,\rho| \le T} \frac{X^\rho}{\rho} + O\left(\frac{X(\log X)^2}{T}\right).$$

Note: This formula shows distribution of primes closely related to location of zeros of $\zeta(s)$.

Pf: Recall that from truncated Perron formula, we have that for $c = 1 + \frac{1}{\log X}$, $2 \le T \le 2X$

$$\psi(x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left(-\frac{\zeta'}{\zeta}(s)\right) X^s \frac{ds}{s} + O\left(\frac{X(\log X)^2}{T}\right)$$

We want to estimate the integral $\frac{1}{2\pi i} \int_{c-iT}^{c+iT} X^s \frac{\zeta'}{\zeta}(s) \frac{ds}{s}$ by moving the line of integration and applying residue theorem. Therefore we need to find the poles of $\frac{\zeta'}{\zeta}(s) \frac{X^s}{s}$ and compute the residues.

X^s has no poles in the complex plane

$\frac{1}{s}$ has a simple pole at 0, no other poles.

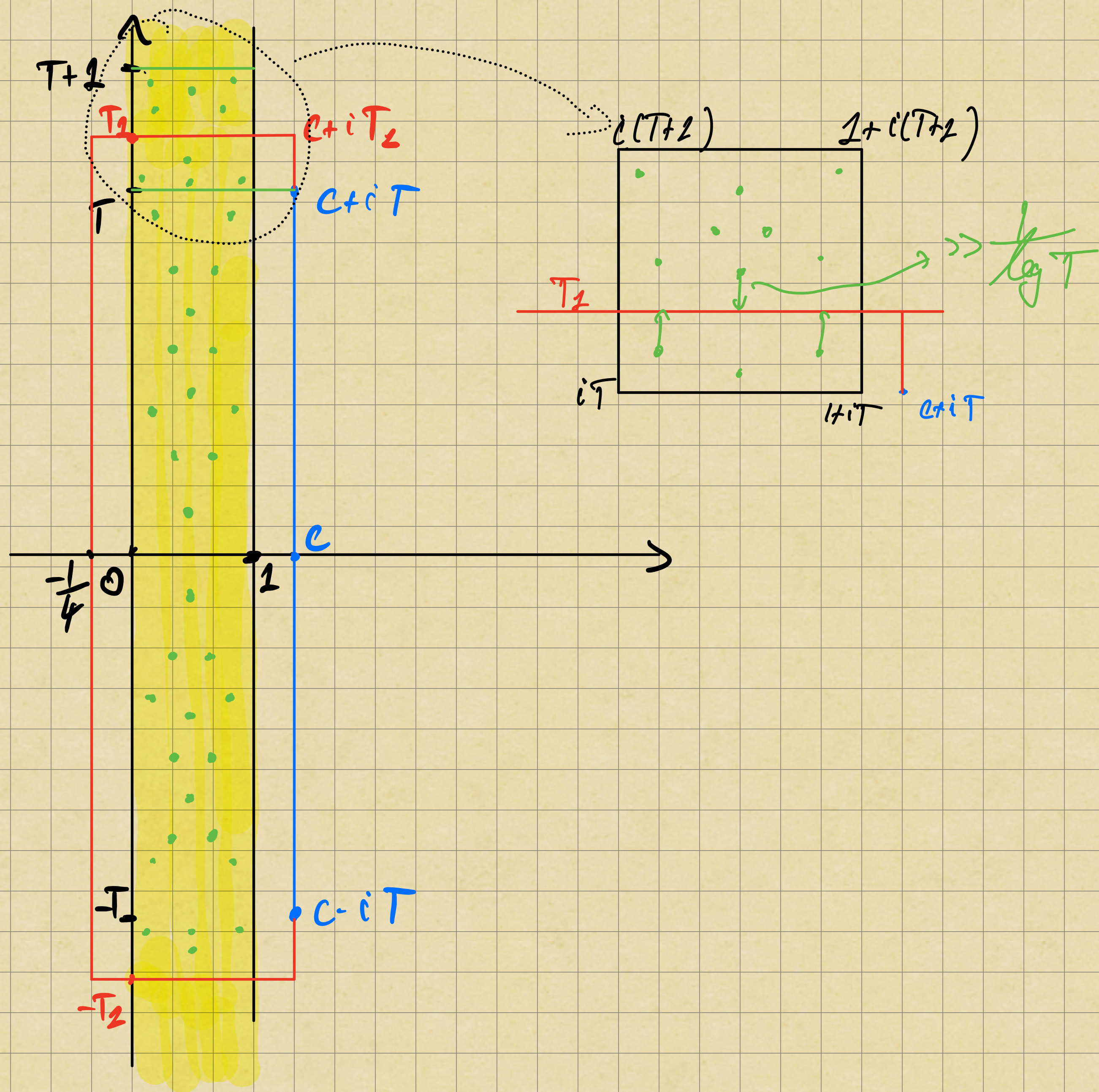
$\frac{\zeta'}{\zeta}(s)$ has a simple pole at 1 and a simple pole at each zero of $\zeta(s)$.

Residue at $s=1$ is $\lim_{s \to 1} (s-1) \frac{\zeta'}{\zeta}(s) \frac{X^s}{s} = -X$

Residue at $s=0$ is $\lim_{s \to 0} X^s \frac{\zeta'}{\zeta}(s) = \frac{\zeta'}{\zeta}(0)$ (a constant)

If $\zeta(s)$ has zero ρ with multiplicity m_ρ,
then $\lim_{s \to \rho} (s-\rho) \frac{\zeta'}{\zeta}(s) \frac{X^s}{s} = m_\rho \frac{X^\rho}{\rho}$.

We want to carefully choose our box of integration such that we avoid poles of $\frac{\zeta'}{\zeta}(s) \frac{X^s}{s}$.



From last time, we know there are $O(\log T)$ zeros of $\zeta(s)$ with $\mathrm{Im}(\rho) \in [T, T+1]$. Therefore there exists $T_1 \in [T, T+1]$ such that all zeros ρ of $\zeta(s)$ satisfy $\big| |\mathrm{Im}\,\rho| - T_1 \big| \gg \frac{1}{\log T}$

Can put absolute values because if ρ is a zero, so is $\bar{\rho}$

We integrate $\frac{x^s}{s}\frac{\zeta'}{\zeta}(s)$ along the box with corners $c-iT_2$, $c+iT_2$, $-\frac{1}{4}+iT_2$ and $-\frac{1}{4}-iT_2$.

This gives

$$\frac{1}{2\pi i}\left(\int_{c-iT_2}^{c+iT_2} x^s\frac{\zeta'}{\zeta}(s)\frac{ds}{s}+\int_{c-iT_2}^{-\frac{1}{4}+iT_2} x^s\frac{\zeta'}{\zeta}(s)\frac{ds}{s}+\int_{-\frac{1}{4}+iT_2}^{-\frac{1}{4}-iT_2} x^s\frac{\zeta'}{\zeta}(s)\frac{ds}{s}+\int_{-\frac{1}{4}+iT_2}^{-c+iT_2} x^s\frac{\zeta'}{\zeta}(s)\frac{ds}{s}\right)=\underbrace{-x}_{\text{residue at }1}+\underbrace{\frac{\zeta'}{\zeta}(0)}_{\text{res at }0}+\underbrace{\sum_{\rho:|\operatorname{Im}\rho|\le T_2}\frac{x^\rho}{\rho}}_{\text{residue at }\rho}.$$

But we know: $\frac{\zeta'}{\zeta}(\sigma+iT_2)=O((\log T)^2)$, for $-\frac{1}{4}\le\sigma\le c$.

Therefore

$$\int_{c+iT_2}^{-\frac{1}{4}+iT_2}\frac{x^s}{s}\frac{\zeta'}{\zeta}(s)\,ds\ll\frac{x^c(\log T)^2}{T}\ll\frac{x(\log x)^2}{T}$$

(and similarly for $\int_{-\frac{1}{4}-iT_2}^{c+iT_2}$).

Now,

$$\int_{-\frac{1}{4}+iT_2}^{-\frac{1}{4}-iT_2}x^s\frac{\zeta'}{\zeta}(s)\frac{ds}{s}\ll\frac{(\log T)^2}{x^{1/4}}\int_{-T_2}^{T_2}\frac{dt}{1+|t|}\ll\frac{(\log T)^3}{x^{1/4}}$$

we used $\frac{\zeta'}{\zeta}(-\frac{1}{4}+it)\ll\log(1+|t|)^2$

This implies $\displaystyle\frac{1}{2\pi i}\int_{c-iT_1}^{c+iT_1} x^s \frac{\zeta'}{\zeta}(s)\frac{ds}{s} = -x + \sum_{\rho : |\operatorname{Im}\rho| \le T_1} \frac{x^\rho}{\rho}$

$$+ O\left(\frac{(\log T)^3}{x^{1/4}} + \frac{x}{T}(\log x)^2\right).$$

Finally, there are $O(\log T)$ zeros ρ with $T \le |\operatorname{Im}\rho| \le T_1$, and each contributes $O\left(\frac{x}{T}\right)$.

Hence $\displaystyle\sum_{\rho : T \le |\operatorname{Im}\rho| \le T_1} \frac{x^\rho}{\rho} \ll \frac{x \log T}{T}$.

Also, $\displaystyle\int_{c+iT}^{c+iT_1} \frac{x^s}{s}\frac{\zeta'}{\zeta}(s)\,ds \ll \frac{x}{T}(\log x)^2$.

(we used $\left|\frac{\zeta'}{\zeta}(c+it)\right| \le (\log x)^2$

Since $c+it$ is at least $\frac{1}{\log x}$ away from a zero ρ, $c = 1 + \frac{1}{\log x}$)

Hence we have indeed

$$\psi(x) = -\frac{1}{2\pi i}\int_{c-iT_1}^{c+iT_1} \frac{x^s}{s}\frac{\zeta'}{\zeta}(s)\,ds + O\left(\frac{x(\log x)^2}{T}\right)$$

$$= x - \sum_{|\operatorname{Im}\rho| \le T_1} \frac{x^\rho}{\rho} + O\left(\frac{x(\log x)^2}{T} + \frac{(\log T)^3}{x^{1/4}}\right)$$

$$= x - \sum_{|\operatorname{Im} \rho| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x (\log x)^2}{T} \right) \qquad \square$$